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Document url: http://www.stanford.edu/~tanqf/gallager_upperbound_1.pdf (old URL)
Current URL: http://www-scf.usc.edu/~qtan/gallager_upperbound_1.pdf

Purpose: Alternative method for finding Gallager's redundancy upper-bound for uniformly distributed alphabet.

Cost (in length) of expanding a node in a tree of D-ary alphabet:

C, cost per node = $\dfrac{(D^m - x)m + xD(m+1)}{D^m + x(D-1)}$ where x is the number of full newly expanded nodes.

$$\frac{dC}{dx} = \frac{[D^m + x(D-1)][-m + D(m+1)] - [(D^m - x)m + xD(m+1)][D-1]}{[D^m + x(D-1)]^2} = \frac{D^{m+1}}{[D^m + x(D-1)]^2}$$

Let n be the number of nodes.

We see that $n = D^m + x(D-1)$. Next we see that the entropy is

$H_D = \log_D[D^m + x(D-1)] = \log_D n$.

We want to maximize the quantity $C - H_D$. We can take the derivative and see that

$$\frac{dC}{dx} = \frac{dH_D}{dx} \Rightarrow \frac{D^{m+1}}{n*^2} = \frac{1}{(\ln D)n*}(D-1) \Rightarrow n* = \frac{\ln D}{D-1}D^{m+1}$$

Having found n*, we can now proceed to find x* as follows:

$$
\begin{aligned}
&x^* \\
&= \frac{n^* - D^m}{D-1} \\
&= \frac{\frac{\ln D}{D-1} D^{m+1} - D^m}{D-1} \\
&= \frac{D^{m+1} \ln D - D^{m+1} + D^m}{(D-1)^2}
\end{aligned}
$$

Next, since we know that $C = m + \frac{xD}{n}$, we can proceed to compute the cost by

$$
\begin{aligned}
&C^* \\
&= m + \frac{x^* D}{n} \\
&= m + \frac{D^{m+1}(D \ln D - D) + D^{m+1}}{(D-1)^2} \frac{(D-1)}{D^{m+1}(\ln D)} \\
&= m + \frac{(D \ln D - D) + 1}{(\ln D)(D-1)}
\end{aligned}
$$

Next, we know that

$$
\begin{aligned}
&H_D^*(n) \\
&= \log_D (D^m + x^*(D-1)) \\
&= \log_D (D^m + \frac{D^{m+1} \ln D - D^{m+1} + D^m}{(D-1)}) \\
&= m + \log_D (1 + \frac{D \ln D - D + 1}{D-1}) \\
&= m + \log_D \frac{D \ln D}{D-1} \\
&= m + 1 + \log_D \ln D - \log_D (D-1)
\end{aligned}
$$

Thus, we see that the redundancy is computed by

$$
\begin{aligned}
&R \\
&= C^* - H_D^* \\
&= \frac{D \ln D - (D-1)}{(\ln D)(D-1)} - 1 - \log_D \ln D + \log_D (D-1) \\
&= \frac{\ln D - (D-1)}{(\ln D)(D-1)} + \log_D \log_D e + \log_D (D-1) \\
&= \frac{1}{D-1} - \log_D e + \log_D (\log_D e) + \log_D (D-1) \\
&= \sigma_D
\end{aligned}
$$

which is Gallager's bound for the general D-ary alphabet case. (QED)