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Document URL: http://www.stanford.edu/~tanqf/lowerbound_1.pdf (old URL)
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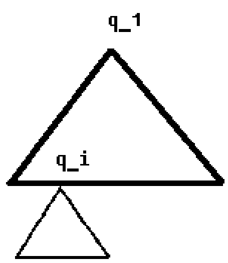
Motivation: Since Gallager upperbounded the Redundancy using P_{max}, probability of most likely source letter, we consider a dual problem of lowerbounding the Redundancy using P_{min}, probability of the least likely source letter.

Diagram:



Denote $tree^{q_i}$ as the tree formed by collapsing everything below the node with probability q_i to just a point. Similarly denote $tree_{q_i}$ as the subtree under the node with probability q_i.

By the law of total probability we have $L = L\left(tree^{q_i}\right) + q_i L(tree_{q_i})$

For the entropy, we can write it as

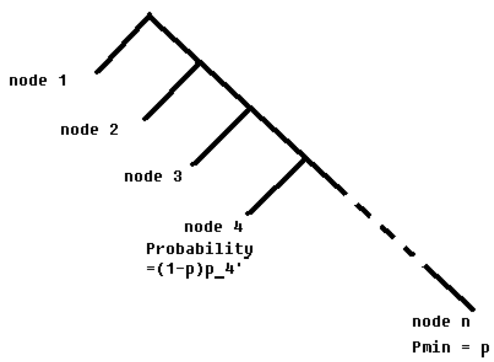
$$\begin{aligned}
&H \\
&= -\sum_k q_k \log q_k \\
&= -\sum_{k \le i} q_k \log q_k - \sum_{k > i} q_k \log q_k \\
&= -\sum_{k \le i} q_k \log q_k - \sum_{k > i} \frac{q_k}{q_i} \log \frac{q_k}{q_i} + q_k \log q_i
\end{aligned}$$

Thus we see that

$$
\begin{aligned}
& R(tree) \\
& = L - H \\
& = L\left(tree^{q_i}\right) + q_i L(tree_{q_i}) - [-\sum_{k \le i} q_k \log q_k - \sum_{k>i} \frac{q_k}{q_i} \log \frac{q_k}{q_i} + q_k \log q_i] \\
& \ge R(tree^{q_i})
\end{aligned}
$$

Thus to find a lower bound, we can collapse the tree to the following form:

Diagram:



Where P_{min} = p. Suppose each leaf occurs with probability $p_i'(1-p)$, where $\sum_{i=1}^{n-1} p_i' = 1$.

We can write the redundancy as

$$
\begin{aligned}
& R \\
& = L - H \\
& = [pn + \sum_{i=1}^{n-1} p_i'(1-p)i] - [-p\log_2 p - \sum_{i=1}^{n-1} p_i'(1-p)\log_2 p_i'(1-p)] \\
& = [pn + \sum_{i=1}^{n-1} p_i'(1-p)i] - [-p\log_2 p - \sum_{i=1}^{n-1} p_i'(1-p)\log_2 p_i' - \sum_{i=1}^{n-1} p_i'(1-p)\log_2(1-p)] \\
& = [pn + \sum_{i=1}^{n-1} p_i'(1-p)i] - [-p\log_2 p - (1-p)\sum_{i=1}^{n-1} p_i'\log_2 p_i' - (1-p)\log_2(1-p)\sum_{i=1}^{n-1} p_i'] \\
& = [pn + \sum_{i=1}^{n-1} p_i'(1-p)i] - [(1-p)H(p_1,..., p_{n-1}) + \mathrm{H}_{binary}(p)] \\
& = pn - \mathrm{H}_{binary}(p) + (1-p)[\sum_{i=1}^{n-1} p_i' i + H(p_1,..., p_{n-1})]
\end{aligned}
$$

Thus we have the following problem:

$$\min_{p_1,\ldots,p_{n-1}'} R \Rightarrow \min_{p_1,\ldots,p_{n-1}'} \sum_{i=1}^{n-1} p_i' i + H(p_1,\ldots,p_{n-1})$$

To incorporate the condition that $\sum_{i=1}^{n-1} p_i' = 1$ we use Lagrange multipliers:

Write $f = \sum_{i=1}^{n-1} p_i' i + H(p_1,\ldots,p_{n-1}) + \lambda(1 - \sum_{i=1}^{n-1} p_i')$.

Note that if we do $\frac{\partial f}{\partial \lambda} = 0 \Rightarrow \sum_{i=1}^{n-1} p_i' = 1$ we get back our original constraint.

By the symmetry of the problem, we need only to take the partial derivative with respect to lets say p_m' for some $0 \le m \le n-1$.

$$\frac{\partial f}{\partial p_m'} = 0$$

$$\Rightarrow m - \log_2 p_m' - \log_2 e - \lambda = 0$$

$$\Rightarrow \log_2 p_m' = m - \log_2 e - \lambda$$

$$\Rightarrow p_m' = 2^{m - \log_2 e - \lambda}$$

Plugging this back into our constraint $\sum_{i=1}^{n-1} p_i' = 1$ we get

$$\sum_{i=1}^{n-1} 2^{i - \log_2 e - \lambda} = 1$$

$$\Rightarrow (2^{-\log_2 e - \lambda}) \sum_{i=1}^{n-1} 2^i = 1$$

$$\Rightarrow (2^{-\log_2 e - \lambda})(2^n - 1) = 1$$

$$\Rightarrow \lambda = \log_2[(2^n - 1)(\frac{1}{e})]$$

Putting this into $p_m' = 2^{m - \log_2 e - \lambda}$ we get

$$p_m' = 2^{m - \log_2 e - \log_2[(2^n - 1)(\frac{1}{e})]} = \frac{2^m}{2^n - 1}.$$

Thus we can see that the distribution that minimizes the redundancy is given by

$$\frac{2}{2^n - 1}(1-p), \frac{2^2}{2^n - 1}(1-p), \ldots, \frac{2^{n-1}}{2^n - 1}(1-p), p$$

Putting this back into $R = pn - \mathrm{H}_{binary}(p) + (1-p)[\sum_{i=1}^{n-1} p_i' i + H(p_1, ..., p_{n-1})]$ we get

$$
\begin{aligned}
R \\
&= pn - \mathrm{H}_{binary}(p) + (1-p)[\sum_{i=1}^{n-1} \frac{2^i}{2^n - 1} i - \sum_{i=1}^{n-1} \frac{2^i}{2^n - 1} \log_2 \frac{2^i}{2^n - 1}] \\
&= pn - \mathrm{H}_{binary}(p) + (1-p)[\sum_{i=1}^{n-1} \frac{2^i}{2^n - 1} i - \sum_{i=1}^{n-1} \frac{2^i}{2^n - 1} i + \log_2 (2^n - 1)] \\
&= pn - \mathrm{H}_{binary}(p) + (1-p) \log_2 (2^n - 1)
\end{aligned}
$$

which is the lower bound we desire.